ANALYSIS OF SINGLE-PARTICLE ENERGY LEVELS OF A NEUTRON IN ^{208}Pb AND LAMBDA IN $^{209}_{\Lambda}$Pb

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Abstract

We calculated single-particle energy of a neutron in ^{208}Pb and single-particle energy of a lambda in $^{209}_{\Lambda}$Pb with harmonic oscillator, Woods-Saxon without spin-orbit and with spin-orbit potentials. In these models, We solved by using Numerove's method to obtain single-particle energy levels in ^{208}Pb and $^{209}_{\Lambda}$Pb. This model can also be extended to include the more complex configurations that arise for the nuclei with nucleon numbers that are in between the magic numbers. The lambda single-energies for 1s,1p,1d,1f states in $^{209}_{\Lambda}$Pb and neutron single-energies for various states in ^{208}Pb are calculated by using Woods-Saxon with spin-orbit potentials. The results of our theoretical calculation for $^{209}_{\Lambda}$Pb are compared with experimental data and with the previous theoretical work. It is observed that our results are nearly agreement with experimental results. Therefore, Numerov's method is suitable to search single particle energy levels.

Keyword: single-particle energies, ^{208}Pb, $^{209}_{\Lambda}$Pb , Numerove method.

Introduction

Single-particle and single-hole neutron states have been previously investigated in the region around the doubly magic ^{208}Pb nucleus [Blomqvist, J., Wahlborn Ark. S]. The interaction of a neutron with the rest of the nucleus referred to as the core has been represented in these studies by a Hamiltonian containing a nuclear Woods-Saxon (WS) potential [Woods . R. D and Saxon. D. S] and a spin-orbit (SO) coupling term. Although the same potential parameterization has been used in [Blomqvist, J., Wahlborn Ark. S] a unique set of parameter values has not been found. They have used the generalized Woods-Saxon (GWS) potential instead of the original WS potential [Woods . R. D and Saxon. D. S] , with the expectation of reproducing the experimental binding energies of single-particle and single-hole neutron orbitals that exist in the neutron shells N = 126 - 184 and 82 - 126, respectively. This potential contains the WS potential plus a term referred to as the surface (SU) potential that maximizes in the nuclear surface and is linearly proportional to the derivative of a WS function. It is a well known fact that the WS potential alone does not reproduce the energies of l= 0 single-particle levels with enough accuracy when applied to a wide nuclide region.

A hypernucleus is a nucleus which contains at least one hyperon in addition to the normal protons and neutrons. The first was discovered by Marian Danysz and Jerzy Pniewski in 1952 using the nuclear emulsion technique.

Protons and neutrons are made of up (u) and down (d) quarks. A Lambda (Λ) hyperon consists of one up, one down, and one strange (s) quark. Like neutrons the Λ-hyperons have no charge, but Λ -s are heavier than neutrons. As a hyperon does not have to obey the Pauli Exclusion principle with the neutrons and protons, it can enter deep inside a nucleus and occupy the same levels already filled with nucleons. This property of hyperons enabled us to view the

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deep-lying shell model structure of nuclei that can not be seen in reactions with nucleons due to the Pauli blocking. The hyperons seem to act as glue inside a nucleus. It was found that if one replaces a neutron with a Λ-hyperon, it makes a bound ${}^{10}_{\Lambda}\mathrm{Li}$ nucleus, while the normal ${}^{10}\mathrm{Li}$ nucleus is known to be unbound [Saha.P.K et al] . This indicates that hypernuclei with large neutron-to-proton ratios could exist in a stable state, even though the corresponding normal neutron-rich nuclei could be unstable.

Although the nucleon-nucleon (NN) interaction is reasonably well known, the ΛN, $\Lambda\Lambda$N and $\Lambda\Lambda$ interactions are yet to be fully understood. One studies the Λ-hypernucleus to estimate the basic Λ-nucleon interaction. Experimentally about thirty-five hypernuclei with one Λ-hyperon and six hypernuclei with two Λ-hyperons have been found so far [Nakazawa. K et al]. The Λ and $\Lambda\Lambda$- separation energies from hypernuclei provide a window to estimate the ΛN, $\Lambda\Lambda$N- interactions properties of Λ and nucleons.

Numerical Calculation

We solved numerically the radial part of Schroedinger's equation by using Numerov's method to obtain neutron single-particle energy. Numerov's method is a numerical method to solve ordinary differential equations of second order in which the first-order term does not appear. It is a fourth-order linear multistep method. The method is implicit, but can be made explicit if the differential equation is linear.

The Numerov method can be used to solve differential equation

$$\frac{d^2u}{dr^2} = -k(r)u(r) + S(r) \qquad (1)$$

Three values of u_{n-1}, u_n, u_{n+1} taken at three equidistant points r_{n-1}, r_n, r_{n+1} are related as follows:

$$u_{n+1}\left(1+\frac{h^2}{12}k_{n+1}\right) = 2u_n\left(1-\frac{5h^2}{12}k_n\right) - u_{n-1}\left(1+\frac{h^2}{12}k_{n-1}\right) + \frac{h^2}{12}(S_{n+1}+10S_n+S_{n-1}) + 0(h^5) \qquad (2)$$

Where $u_n = u(r_n)$, $k_n = k(r_n)$, $S_n = S(r_n)$ and $h = r_{n+1}-r_n$.

For nonlinear equations of the form

$$\frac{d^2u}{dr^2} = f(u,r) \qquad (3)$$

the method gives

$$u_{n+1}-2u_n + u_{n-1} = \frac{h^2}{12}(f_{n+1}+10f_n+f_{n-1}) + 0(h^5) \qquad (4)$$

This is an implicit linear multistep method which reduces to the explicit method given above if f is linear in u by setting f (u, r) = -k(r)u(r) + S(r). It achieves order-4 accuracy.

In numerical physics the method is used to find solutions of the unidimensional Schrodinger equation for arbitrary potentials. It is used to solve the radial equation for a spherically symmetric potential. After separating the variables and analytically solving the angular equation, we get the following equation of the radial function R(r):

$$\frac{d^2u(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right]u(r)=0 \quad (5)$$

Where $u(r) = rR_{nl}$ is the reduced radial wave function.

A regular solution near the origin for $u(r): u(r\to 0)\to r^{\ell+1}$

The asymptotic solution at $r\to\infty: u(r\to\infty)\to u(r)=e^{-\alpha r^2}$, α = constant

The Schrödinger Radial Equation can be written as follow:

$$\frac{d^2u(r)}{dr^2}+k(r)u(r)=0 \quad (6)$$

$$\frac{d^2u(r)}{dr^2}=u_r''=-k(r)u(r) \quad (7)$$

$k(r) = \frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right]$ is the kernel equation.

Equation (5) can be solved by Numerov Algorithm as follow:

First we split the r range into N points according to $r_n = r_{n-1} + h$; then we write the wave function $u_n \equiv u(r_n)=u(r_{n-1}+h)$ and $k_n \equiv k(r_n) =k(r_{n-1}+h)$.

Expanding u(r) around r_n:

$$u(r) = u(r_n) + (r-r_n)u'(r_n) + \frac{(r-r_n)^2}{2!}u''(r_n) + \frac{(r-r_n)^3}{3!}u'''(r_n)$$

$$+\frac{(r-r_n)^4}{4!}u^{iv}(r_n) + 0(h^5) \quad (8)$$

If we evenly discretize the space, we get a grid of r points, where $h=r_{n+1}-r_n$. By applying the above equations to this discreet space, we get a relation between the u_n and u_{n+1}:

$$u_{n+1} \equiv u(r_n+h) = u(r_n) + hu'(r_n) + \frac{h^2}{2!}u''(r_n) + \frac{h^3}{3!}u'''(r_n) + \frac{h^4}{4!}u^{iv}(r_n) + 0(h^5) \quad (9)$$

Since $u_n \equiv u(r_n)$,

$$u_{n+1} = u_n + hu_n' + \frac{h^2}{2}u_n'' + \frac{h^3}{6}u_n''' + \frac{h^4}{24}u_n^{iv} + 0(h^5) \quad (10)$$

Computationally, this amounts to taking a step forward by an amount h. If we want to take a step backwards, we repleace every h with –h and get the expression for u_{n-1}:

$$u_{n-1} \equiv u(r_n-h) = u(r_n) - hu'(r_n) + \frac{h^2}{2!}u''(r_n) - \frac{h^3}{3!}u'''(r_n) + \frac{h^4}{4!}u^{iv}(r_n) + 0(h^5) \quad (11)$$

$$u_{n-1} = u_n - hu_n^{'} + \frac{h^2}{2}u_n^{''} + \frac{h^3}{6}u_n^{'''} + \frac{h^4}{24}u_n^{iv} + 0(h^5) \tag{12}$$

By summing the two equation (10) and equation (12),

$$u_{n+1} - 2u_n + u_{n-1} = h^2 u_n^{''} + \frac{h^4}{12}u_n^{iv} + 0(h^5) \tag{13}$$

To get an expression for the u_n^{iv} factor, we simply have to differentiate $u_n^{''} = -k_n u_n$ twice and approximate it again in the same way we did this above:

$$u_n^{iv} = \frac{d^2}{dx^2}(-k_n u_n) \tag{14}$$

From equation (13),

$$u_{n+1} - 2u_n + u_{n-1} = h^2 u_n^{''} \tag{15}$$

$$u_{n+1}^{''} - 2u_n^{''} + u_{n-1}^{''} = h^2 u_n^{iv} \tag{16}$$

$$-u_{n+1}k_{n+1} + 2u_n k_n - u_{n-1}k_{n-1} = h^2 u_n^{iv} \tag{17}$$

Substituting equation (17) into equation (13)

$$u_{n+1} - 2u_n + u_{n-1} = -h^2 u_n k_n + \frac{h^2}{12}(-u_{n+1}k_{n+1} + 2u_n k_n - u_{n-1}k_{n-1}) \tag{18}$$

$$u_{n+1}(1 + \frac{h^2}{12}k_{n+1}) - 2u_n(1 - \frac{5h^2}{r^2}k_n) + u_{n-1}(1 + \frac{h^2}{12}k_{n-1}) = 0 \tag{19}$$

from equation (19), we obtained the following relations.

For forward recursive relation

$$u_n = \frac{2(1 - \frac{5h^2}{12}k_{n-1})u_{n-1} - (1 + \frac{h^2}{12}k_{n-2})u_{n-2}}{(1 + \frac{h^2}{12}k_n)} \tag{20}$$

For backward recursive relation

$$u_{n-1} = \frac{2(1 - \frac{5h^2}{12}k_n)u_n - (1 + \frac{h^2}{12}k_{n+1})u_{n+1}}{(1 + \frac{h^2}{12}k_{n-1})} \tag{21}$$

Therefore when we calculate our wave function using the backward-forward technique, we should note that the recursive formulas imply having knowledge of two initial values for each direction. It is also necessary to know the first derivative at the appropriate order. By subtraction from equation (20) to equation (21)

$$u_{n+1} - u_{n-1} = 2hu_n^{'} + 2\frac{h^3}{6}u_n^{'''} \tag{22}$$

$$u_n^{'} = \frac{1}{2h}\left[(1+\frac{h^2}{6}k_{n+1})u_{n+1} - (1+\frac{h^2}{6}k_{n-1})u_{n-1}\right] \tag{23}$$

We used harmonic-oscillator potential and Woods-Saxon potential in Schrodinger equation to obtain wave functions and energy states. The harmonic-oscillator potential (HO) is given by

$$V(r) = \frac{1}{2}\mu\omega^2 r^2 \tag{24}$$

where μ is reduce mass of nucleon and ω is the parameter. $\hbar\omega \approx 45 \times A^{-1/3}$.

The Woods-Saxon potential is based upon the sum of a spin-independent central potential, a spin-orbit potential, and the Coulomb potential. The spin-orbit potential has the form, $V_{so}(r)\ell.s$, where ℓ is the orbital angular momentum and s is the intrinsic spin angular momentum of the nucleon.

$$V(r) = V_0(r) + V_{so}(r)\vec{\ell}.\vec{s} + V_c(r) \tag{25}$$

$V_0(r)$ is the spin-independent central potential:

$$V_0(r) = \frac{V_0}{1+\exp\left(\frac{r-R}{a}\right)}, \quad R = \text{nuclear radius} \tag{26}$$

$R = r_0 A^{1/3}$. $V_{so}(r)$ is the spin-orbit potential:

$$V_{so}(r) = V_{so}\frac{1}{r}\frac{d}{dr}\frac{1}{1+\exp\left(\frac{r-R}{a}\right)} = -V_{so}\frac{1}{ra}\frac{\exp\left(\frac{r-R}{a}\right)}{\left(1+\exp\left(\frac{r-R}{a}\right)\right)^2} \tag{27}$$

The Woods-Saxon form of potential is assumed for the single-particle potential for the nucleon. The chosen parameters are r_0= 1.25 fm and a= 0.65 fm .

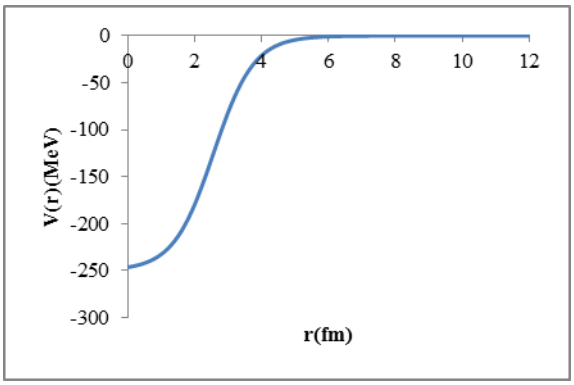

Figure 1 Woods-Saxon potential

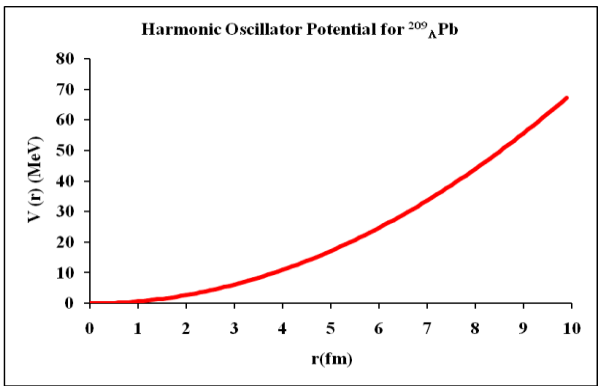

Figure 2 Harmonic-oscillator potential

Calculation of Energy Levels

Since both $u_{out}(r)$ and $u_{in}(r)$ satisfy a homogeneous equation, their normalization can always be chosen so that they are set to be equal at the r_c point. An energy eigen value is then signaled by the equality of derivatives at this point. At the matching point the eigen functions $u_{out}(r)$ and $u_{in}(r)$ and first derivatives $u'_{out}(r)$ and $u'_{in}(r)$ must all satisfy the continuity conditions:

$$(u_{out})_{rc} = (u_{in})_{rc} \qquad (u'_{out})_{rc} = (u'_{in})_{rc} \tag{28}$$

thus, we can write the corresponding condition for the logarithmic derivative at r_c as

$$\left[\frac{u'_{out}}{u_{out}}\right]_{rc} = \left[\frac{u'_{in}}{u_{in}}\right]_{rc} \tag{29}$$

and then we can define a Match (E) function at r_c whose zeros correspond to the energy eigenvalues as

$$\text{Match(E)} = \left[\frac{u'_{out}}{u_{out}}\right]_{rc} - \left[\frac{u'_{in}}{u_{in}}\right]_{rc} \tag{30}$$

Therefore we proceed numerically in the following way: we set a trial energy range splitting this E range into N points, according to $E_n = E_{n-1} + \Delta E$, where ΔE is the energy step. For each E_n we calculate their eigenfunctions u_{out} and u_{in} at the r_c point; and we build the Match(E) function here, looking for a change of sign in it (which implies a zero cross).

When we find the energy eigenvalue, the calculated inwards and outwards eigenfunctions will tend not to match at the r_c point. However we can look for a strategy to solve this problem. Denoting the outwards and inwards functions directly obtained from the recursive formulas as uo(r) and ui (r), respectively.

Results and Discussion

4.1 Single-Particle Energy Levels of a Neutron in ^{208}Pb

We solved numerically the shoredinger equation by using Numerov's method to obtain single particle energy levels of a neutron in ^{208}Pb. It is used the harmonic oscillator potential, central potential and Woods-Saxon potential. The neutron single particle energy levels and sub energy levels are obtained from central potential and Woods-Saxon potential with spin orbit. The results are shown in table(1) and figure (3). It is observed that energy levels are split sub energy levels by using Woods-Saxon potential. The wave functions for various state such as s, p and d states. The results are shown in Fig. (4), (5) and (6). It is seen that wave functions for all bound states are convergent. Figure (7), (8) and (9) are shown various principal number of s, p and d state. These wave functions are shifted to higher state.

Table 1 Single particle energies in ^{208}Pb for various states

Harmonic Oscillator Potential		**Woods-Saxon potential without Spin-orbit**		**Woods-Saxon potential with Spin-orbit**	
Shell	Energy (MeV)	Shell	Energy (MeV)	Shell	Energy (MeV)
1s	-42.6	1s	-42.6	1s	-42.6
1p	-34.76	1p	-38.36	$1p_{3/2}$ $1p_{1/2}$	-38.45 -38.19
1d, 2s	-27.16	1d 2s	-33.41 -31.6	$1d_{5/2}$ $1d_{3/2}$ 2s	-33.68 -33.03 -31.6
1f , 2p	-19.56	1f 2p	-27.72 -24.94	$1f_{7/2}$ $1f_{5/2}$ $2p_{3/2}$ $2p_{1/2}$	-28.26 -27.04 -25.12 -24.59
1g, 2d, 3s	-11.96	1g 2d 3s	-21.37 -17.85 -14.44	$1g_{9/2}$ $1g_{7/2}$ $2d_{5/2}$ $2d_{3/2}$	-22.28 -20.3 -18.28 -17.23
1h, 2f, 3p	-4.36	1h 2f 3p	-11.96 -10.44 -8.66	$1h_{11/2}$ $1h_{9/2}$ $2f_{7/2}$ $2f_{5/2}$	-15.79 -12.9 -11.15 -9.48

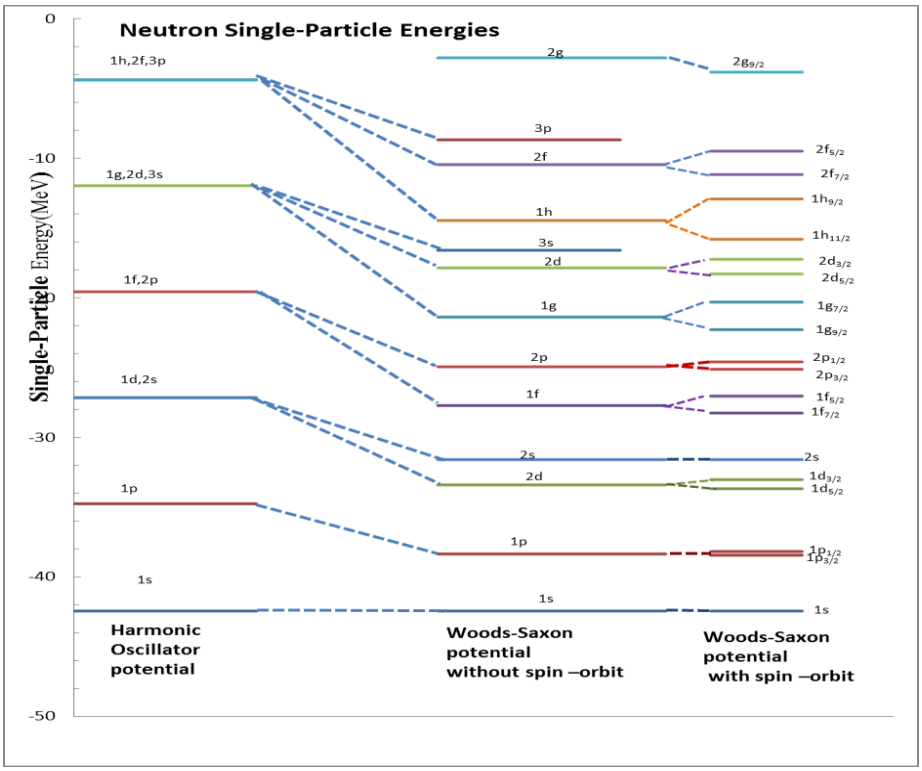

Figure 3 Neutron single-particle states in ^{208}Pb

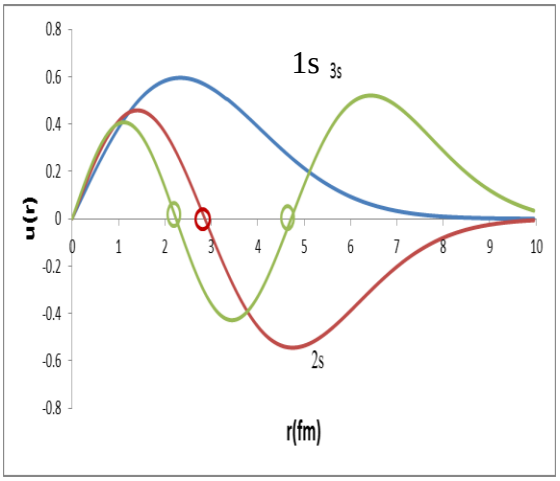

Figure 4 s-states wave functions for ^{208}Pb

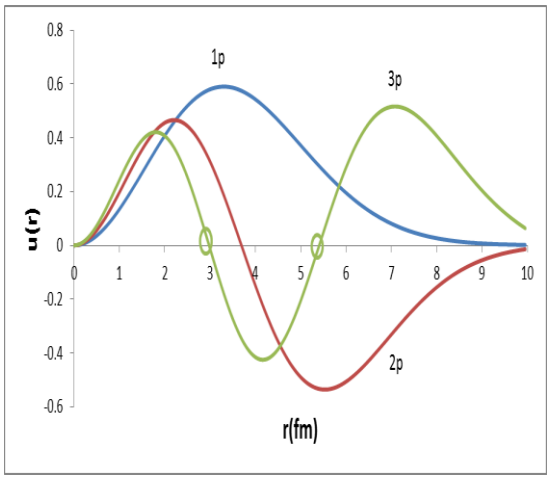

Figure 5 p-states wave functions for ^{208}Pb

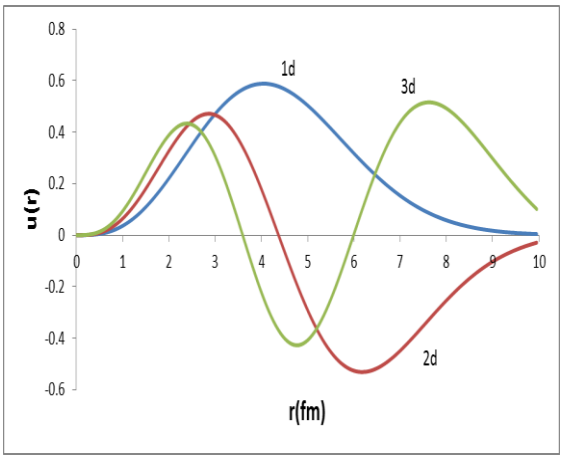

Figure 6 d-states wave functions for ^{208}Pb

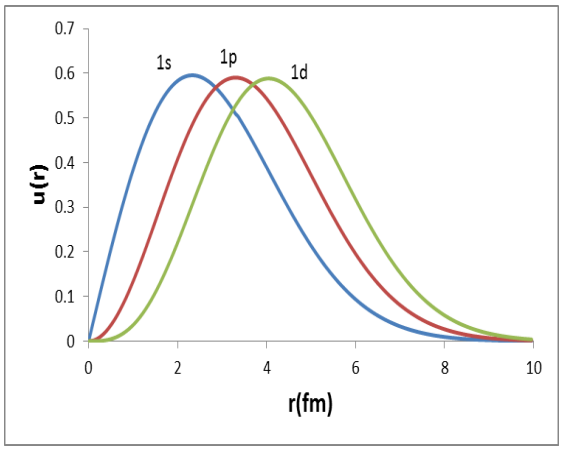

Figure 7 wave functions for 1s,1p and 1d states

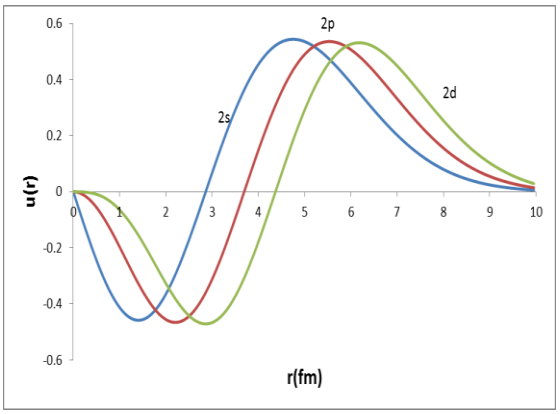

Figure 8 wave functions for 2s, 2p and 2d for ^{208}Pb

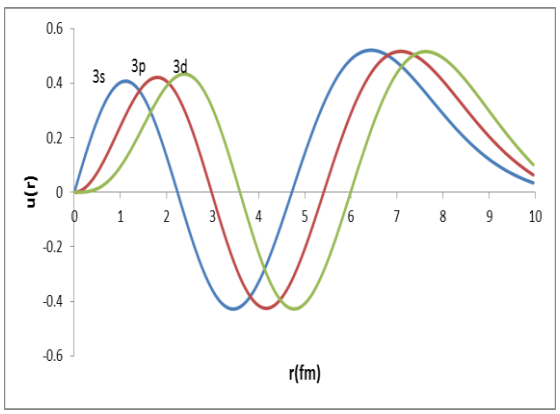

Figure 9 wave functions for 3s,3p and 3d for ^{208}Pb

4.2 Single-Particle Energy Levels of a Lambda in $^{209}_{\Lambda}Pb$

The number of neutrons are allowed by the Pauli principle to occupy one of these levels. As a hyperon does not have to obey the Pauli Exclusion principle with the neutrons and protons, it can enter deep inside a nucleus and occupy the same levels already filled with nucleons. Lambda single-particle energy in $^{209}_{\Lambda}Pb$ are calculated by using Numerov's method. The results are shown in the following figures.

Fig.(10) shows lambda s-states wave functions with harmonic oscillator potential for $^{209}_{\Lambda}Pb$. Fig. (11) and (11) show lambda p and d-states wave functions with harmonic oscillator potential for $^{209}_{\Lambda}Pb$. It is seen that all wave functions are finite. The lambda single-particle states in $^{209}_{\Lambda}Pb$ with Woods-Saxon spin orbit potential are obtained by solving Schrödinger equation. The results are shown in Table (1). It is indicated the effect of the spin-orbit potential in splitting the states of a given ℓ value. The overall strength of the spin-orbit potential has been determined empirically. We compare our results with experimental results [Ajimura. S et al] and theoretical results of *Vidana.A et al*.

Table 2 Comparison of theoretical results and experimental results of Lambda single-particle energies in $^{209}_{\Lambda}Pb$

Sub Shell	Theoretical results [*Vidana.A et al*] (MeV)			Experimental results [Ajimura.S et al] (MeV)	Our results (MeV)
	O	A	F		
1s	-23.1	-29.5	-26.5	-27.0	-25.85
1p	-19.6	-25.7	-22.4	-22.0	-21.89
1d	-14.5	-21.0	-17.5	-17.0	-17.18
1f	-10.5	-15.7	-11.8	-12.0	-11.9
1g	-5.1	-9.7	-5.6	-7.0	-6.0

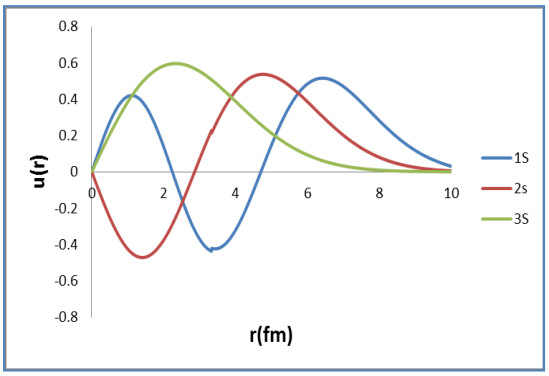

Figure 10 Neutron s-states wave functions for $^{209}_{\Lambda}Pb$

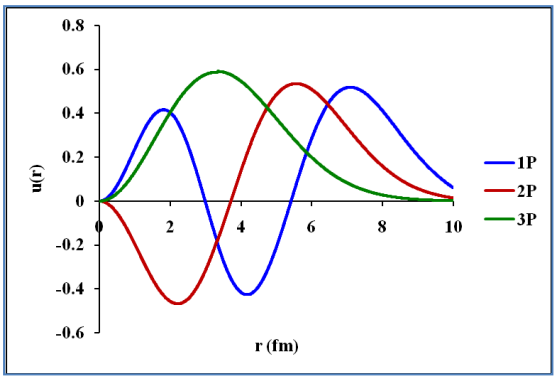

Figure 11 Neutron p-states wave functions for $^{209}_{\Lambda}Pb$

Conclusion

We calculated neutron single particle energies in ^{208}Pb and lambda single particle energies in $^{209}_{\Lambda}$Pb by using Numerov's method for harmonic oscillator, central potential and spin-orbit WoodsSaxon potential. It is observed that the energy levels are split in to sub energy levels. Our results for $^{209}_{\Lambda}$Pb are compared with experimental results [Ajimura.S et al] and other theoretical results by Vidana. A. et al., are calculated Brueckner-Hartree-Fock method for Nijmegen various O, A and F which are divided according to effective mass of lambda $m^{*}_{\Lambda} / m_{\Lambda} = 0.78, 0.67$ and 0.81. Our calculated results for $^{209}_{\Lambda}$Pb are nearly agreement with the experimental results and theoretical results of Nijmegen potential F.

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